Hartshorne chapter 2 section 2 solutions

Jack Ceroni
(Dated: Tuesday 23rd June, 2026)

Contents

I. Introduction

Current solution tally: 9/19

II. Solutions

A. Problem 2.2.1

To produce such an isomorphism, we obviously need a homeomorphism between the underlying topological spaces $\phi : D(f) \to \text{Spec}(A_f)$ and an isomorphism of sheaves $\phi^{\#} : \mathcal{O} \to \phi_* \mathcal{O}_X|_{D(f)}$ where $\mathcal{O}$ is the structure sheaf (such that the morphism induces a local homomorphism of stalks).

If you look in Sec. III, you will see that we have already constructed a bijection between $D(f)$ (the collection of all prime ideals which do not contain (f), hence all prime ideals which do not intersect it) and $\text{Spec}(A_f)$ (the prime ideals of the localization A_f). We denote this map by ϕ (i.e. $\phi : D(f) \to \text{Spec}(A_f)$ is given by $\phi(\mathfrak{p}) = S^{-1}\mathfrak{p}$ and $\phi^{-1}(\mathfrak{q}) = \rho^{-1}(\mathfrak{q})$, where $\rho : A \to S^{-1}A$ is the localization map with $S = \{1, f, f^2, \dots\}$). Our first claim is that $\phi^{-1}(V(\mathfrak{a})) = D(f) \cap V(\rho^{-1}(\mathfrak{a}))$. Indeed, note that if $\mathfrak{p}$ is a prime which contains $\mathfrak{a}$, then $\phi^{-1}(\mathfrak{p}) = \rho^{-1}(\mathfrak{p})$ is a prime which does not contain f and which contains $\rho^{-1}(\mathfrak{a})$. Similarly, if $\mathfrak{p}$ is a prime containing $\rho^{-1}(\mathfrak{a})$ which does not contain f then $\phi(\mathfrak{p})$ is prime and contains $(\phi \circ \rho^{-1})(\mathfrak{a}) = \mathfrak{a}$ (see Rem. III.1), so $\mathfrak{p} \in \phi^{-1}(V(\mathfrak{a}))$, and we have the desired equality. This implies that ϕ is continuous.

On the other hand, we claim that $\phi(V(\mathfrak{a}) \cap D(f)) = V(\phi(\mathfrak{a}))$. If $\mathfrak{p}$ is prime, contains $\mathfrak{a}$, and does not contain (f), then $\phi(\mathfrak{p})$ is prime and contains $\phi(\mathfrak{a})$. On the other hand, if $\mathfrak{p}$ is prime and contains $\phi(\mathfrak{a})$, then $\phi^{-1}(\mathfrak{p})$ is prime, doesn't contain f, and contains $(\rho^{-1} \circ \phi)(\mathfrak{a})$, which contains $\mathfrak{a}$, so $\mathfrak{p} \in \phi(V(\mathfrak{a}) \cap D(f))$. Thus, ϕ is in fact a homeomorphism.

We still need to produce an isomorphism of sheaves. We define $\phi^{\#}(U) : \mathcal{O}(U) \to \mathcal{O}_X(\phi^{-1}(U))$ as follows. Given a section $s : U \to \bigsqcup_{\phi(\mathfrak{p})\in U}(A_f)_{\phi(\mathfrak{p})}$ in $\mathcal{O}(U)$, we can define function $\widetilde{s} : \phi^{-1}(U) \to \bigsqcup_{\mathfrak{p}\in\phi^{-1}(U)} A_{\mathfrak{p}}$ as $\widetilde{s} = \Phi_U^{-1} \circ s \circ \phi|_{\varphi^{-1}(U)}$, where $\Phi_U : \bigsqcup_{\mathfrak{p}\in\phi^{-1}(U)} A_{\mathfrak{p}} \to \bigsqcup_{\phi(\mathfrak{p})\in U}(A_f)_{\phi(\mathfrak{p})}$ is the unique map extending all of the isomorphisms $\Phi_{\mathfrak{p}} : A_{\mathfrak{p}} \to (A_f)_{\phi(\mathfrak{p})}$ introduced in Cor. III.0.1, for all $\mathfrak{p} \in \phi^{-1}(U)$. This also gives us a candidate for $(\phi^{\#}(U))^{-1}$: the map taking $\widetilde{s}$ to $\Phi_U \circ s \circ (\phi|_{\varphi^{-1}(U)})^{-1}$. The fact that both of these maps are homomorphisms simply follows from the fact that we can add and multiply functions, and that Φ is a homomorphism when restricted to the individual rings. For example,

$$(\Phi^{-1} \circ s_1 \cdot s_2 \circ \phi)(\mathfrak{p}) = \Phi^{-1}(s_1(\phi(\mathfrak{p})) \cdot s_2(\phi(\mathfrak{p}))) = \Phi^{-1}(s_1(\phi(\mathfrak{p}))) \cdot \Phi^{-1}(s_2(\phi(\mathfrak{p}))) \tag{1}$$

To prove that this function is, in fact, an element of $\mathcal{O}_X(\phi^{-1}(U))$, first note that

$$\widetilde{s}(\mathfrak{p}) = (\Phi^{-1} \circ s \circ \phi)(\mathfrak{p}) = (\Phi^{-1} \circ s)(\phi(\mathfrak{p})) \in \Phi^{-1}((A_f)_{\phi(\mathfrak{p})}) = A_{\mathfrak{p}} \tag{2}$$

so we satisfy the first condition. We also require that for each $\mathfrak{p} \in \phi^{-1}(U)$, there exists a neighbourhood $\mathfrak{p} \in V \subset \phi^{-1}(U)$ and $a, b \in A$ such that $\widetilde{s}(\mathfrak{q}) = \frac{a}{b} \in A_{\mathfrak{q}}$ for all $\mathfrak{q} \in V$ (in particular, $b \notin \mathfrak{q}$ for any $\mathfrak{q} \in V$). It is of course the case that given some $\phi(\mathfrak{p}) \in U$, we can produce a neighbourhood V such that $s = \frac{a'/f^n}{b'/f^m}$ on V. It then follows that on $\phi^{-1}(V) \subset \phi^{-1}(U)$, which contains $\mathfrak{p}$, we have

$$(\Phi^{-1} \circ s \circ \phi)(\mathfrak{q}) = \Phi^{-1}\left(\frac{a'/f^n}{b'/f^m}\right) = \frac{a' f^m}{b' f^n} \in A_{\mathfrak{q}} \tag{3}$$

where $b' f^n \notin \mathfrak{q} \in \phi^{-1}(V)$, as if it were, then either $b' \in \mathfrak{q}$ or $f \in \mathfrak{q}$. The latter cannot happen by definition of $\mathfrak{q}$. If the former did happen, then we would have $\frac{b'}{f^m} \in \phi(\mathfrak{q}) \in V$, which is not the case. Thus, $\widetilde{s}$ is a valid element of the structure sheaf. Showing that the inverse homomorphism also outputs a valid element of the structure sheaf follows a similar procedure.

We now need to show that $\phi^{\#}$ is actually a natural transformation, which is to say that it behaves nicely with respect to restriction. Suppose U and V are open subsets of $\mathrm{Spec}(A_f)$, with $V \subset U$. We need to show that $\phi^{\#}(s|_V) = \phi^{\#}(s)|_{\phi^{-1}(V)}$, with $s \in \mathcal{O}(U)$. This more or less follows from the definitions:

$$\phi^{\#}(s|_V) = \Phi_V^{-1} \circ s|_V \circ \phi|_{\phi^{-1}(V)} = (\Phi_U^{-1} \circ s \circ \phi)|_{\phi^{-1}(V)} = \phi^{\#}(s)|_{\phi^{-1}(V)} \tag{4}$$

The same logic applies to the inverse mapping. Thus, we have shown that as locally ringed spaces, $(D(f), O_X|_{D(f)}) \simeq \mathrm{Spec}(A_f)$, as desired.

B. Problem 2.2.2

First consider the case where $X = \mathrm{Spec}(A)$, and $\mathcal{O}_X$ is the structure sheaf, so it is an affine scheme. If $U \subset X$ is an open set, then $X - U = V(\mathfrak{a})$ for some ideal $\mathfrak{a} \subset A$, which has generating set $(f_\alpha)_{\alpha\in J}$. It follows that $V(\mathfrak{a}) = \bigcap_{\alpha\in J} V(f_\alpha)$, so $U = \bigcup_{\alpha\in J} D(f_\alpha)$. In other words, we can cover U with open sets of the form $D(f)$. We know that $(D(f), \mathcal{O}_X|_{D(f)})$ is an affine scheme from Problem 2.2.1, so it follows that we can cover $(U, \mathcal{O}_X|_U)$ in affine schemes, which implies it is a scheme (note that restricting sheaves to open sets behaves nicely, so restricting to U and then restricting to $D(f)$ is the same as restricting to $D(f)$ from $\mathcal{O}_X$).

To prove the general case, note that if U is an open set of generic scheme X, then we can cover X in open sets which are isomorphic to affine schemes, $(V_\alpha, \mathcal{O}_X|_{V_\alpha})$. It then follows that $(U \cap V_\alpha, \mathcal{O}_X|_{U\cap V_\alpha})$ is an open subset of an affine scheme, which means it is a scheme, which means we can cover it in affine schemes $(W_{\alpha,\beta}, \mathcal{O}_X|_{W_{\alpha,\beta}})$. Taking all of these affine schemes will cover $(U, \mathcal{O}_X|_U)$, which implies it is itself a scheme.

C. Problem 2.2.3

Part A. Suppose $s \in \mathcal{O}_X(U)$ is nilpontent, so that $s^N = 0$ for some N. Since restriction respects ring structure, $(s|_V)^N = (s^N)|_V = 0$ for some $V \subset U$, so the stalk around any $p \in U$ will have a nilpotent element, namely the germ s_p. On the other hand, suppose stalk $\mathcal{O}_{X,p}$ has nilpotent germ s_p, so $s_p^N = [s,U]^N = [s^N, U] = 0$ for some N. The 0-germ 0 at p is simply $[0,V]$, for some open set V around p, so we must have s^N and 0 agreeing on some neighbourhood $W \subset U \cap V$ of p. Therefore, $\mathcal{O}_X(W)$ contains a nilpotent element.

Part B. Recall that the stalks of the sheafification $\mathcal{O}^{\#}_{X,\text{red}}$ are precisely the stalks of the presheaf $U \mapsto \mathcal{O}_X(U)_{\text{red}}$, which have no nilpotent elements. Therefore, if we prove that $(X, \mathcal{O}^{\#}_{X,\text{red}})$ is a scheme, it will be a reduced scheme.

We will begin by assuming that $(X, \mathcal{O}_X) = (\text{Spec}(A), \mathcal{O})$. The idea is to show that $(\text{Spec}(A), (\mathcal{O}^{\#}_{\text{red}}) \simeq (\text{Spec}(A/\mathfrak{r}), \mathcal{O}')$, where $\mathfrak{r} = \text{Rad}(0)$, the nilradical of A. For any ideal $\mathfrak{a}$, we have the quotient map $\pi : A \to A/\mathfrak{a}$. This means that we have an induced map of locally ringed spaces $(f, f^{\#}) : (\text{Spec}(A/\mathfrak{a}), \mathcal{O}') \to (\text{Spec}(A), \mathcal{O})$, where $f(\mathfrak{p}) = \pi^{-1}(\mathfrak{p})$. If we restrict the image of f to $V(\mathfrak{a})$, we actually find that $f : \text{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a})$ is a homeomorphism. The bijective correspondence between the prime ideal of $A/\mathfrak{a}$ and the prime ideal of A which contain $\mathfrak{a}$ is obvious. Continuity is obvious from the fact that π is induced by the quotient homomorphism $A \to A/\mathfrak{a}$. It is also easy to see that for $\mathfrak{a} \subset \mathfrak{b}$, we have $\pi(V(\mathfrak{b})) = V(\pi(\mathfrak{b}))$. Therefore,

$$\text{Spec}(A/\mathfrak{r}) \simeq V(\mathfrak{r}) = V((0)) = \text{Spec}(A) \tag{5}$$

as desired: $f : \text{Spec}(A/\mathfrak{r}) \to \text{Spec}(A)$ is a homeomorphism.

The other thing we need to do is prove that the sheaves $\mathcal{O}^{\#}_{\text{red}}$ and $f_*\mathcal{O}'$ are isomorphic. Of course, $f^{\#} : \mathcal{O} \to f_*\mathcal{O}'$ is a local morphism of sheaves. Note that if $s \in \mathcal{O}(U)$, then $f^{\#}(U)(s) = \pi' \circ s \circ f = 0$ where $\pi'|_{A_\mathfrak{p}} : A_\mathfrak{p} \to (A/\mathfrak{r})_{\pi(\mathfrak{p})}$ is the reduction map, where we recall that localization commutes with quotients. If s is nilpotent, then $s(\mathfrak{p}) \in \mathfrak{r}_\mathfrak{p}$ for every $\mathfrak{p} \in U$, so $f^{\#}(U)(s) = 0$. It follows that $f^{\#}(U) : \mathcal{O}(U) \to \mathcal{O}'(f^{-1}(U))$ descends to a morphism $f^{\#}_{\text{red}}(U) : \mathcal{O}(U)_{\text{red}} \to \mathcal{O}'(f^{-1}(U))$. Moreover, these morphisms clearly form a local morphism of presehaves. It follows that it extends a unique morphism of sheaves $g : \mathcal{O}^{\#}_{\text{red}} \to f_*\mathcal{O}'$. We want to show that this map is an isomorphism of sheaves, so we show that it induces an isomorphism on the stalks. Indeed, we have $\mathcal{O}^{\#}_{\text{red}, f(p)} = \mathcal{O}_{\text{red}, f(p)}$. $g_{f(p)}$ takes germs $[V, [s]]$ to germ $[V, \pi' \circ s \circ f]$, where $[s]$ is an equivalence class in $\mathcal{O}(V)_{\text{red}}$. If $\pi' \circ s \circ f = 0$ on V, then we must have $s(f(\mathfrak{p}))$ nilpotent for every $\mathfrak{p} \in f^{-1}(V)$, hence $s(\mathfrak{p})$ is nilpotent for every $\mathfrak{p} \in V$, so $[s] = 0$, and we have injectivity. Surjectivity follows from the fact that π' is surjective and f is a homeomorphism. Thus, we have our isomorphism of stalks, and the map as an isomorphism of sheaves.

To prove the general case, we just apply the result above locally. In particular, if $(X, \mathcal{O}_X)$ is a scheme, then locally, $(U, \mathcal{O}_X|_U)$ is isomorphic to $(\text{Spec}(A), \mathcal{O})$. Thus, $(U, \mathcal{O}_X|^{\#}_{U,\text{red}})$ is locally isomorphic to $(\text{Spec}(A), \mathcal{O}^{\#}_{\text{red}})$, which is isomorphic to $(\text{Spec}(A/\mathfrak{r}), \mathcal{O}')$, which is itself an affine scheme.

To prove the final thing, let $\text{red} : \mathcal{O}_X \to \mathcal{O}^{\#}_{X,\text{red}}$ be the unique morphism of sheaves which extends the morphism of presheaves $\text{red} : \mathcal{O}_X \to \mathcal{O}_{X,\text{red}}$ which takes $s \in \mathcal{O}_X(U)$ to its reduction $[s] \in \mathcal{O}_X(U)_{\text{red}}$. The morphism $(\text{id}, \text{red}) : X_{\text{red}} \to X$ is a morphism of schemes which is a homeomorphism on the underlying topological spaces.

Part C. Of course, if we write $f = (h, h^{\#})$, then g will be the mapping h on the underlying topological spaces. We require the existence of unique $g^{\#} : \mathcal{O}^{\#}_{Y,\text{red}} \to h_*\mathcal{O}_X$ such that $g^{\#} \circ \text{red} = h^{\#}$. If we can prove the existence of a unique $\chi : \mathcal{O}_{Y,\text{red}} \to h_*\mathcal{O}_X$ (morphism of presheaves) where $\chi \circ \text{red} = h^{\#}$ as presheaf morphisms, where red in this context is the presheaf morphism taking a section to its reduction, then by uniqueness of extension to the sheafification, $g^{\#}$ is the unique extension of χ. The idea is to simply define $\chi(V) : \mathcal{O}_Y(V)_{\text{red}} \to \mathcal{O}_X(h^{-1}(V))$ as $\chi(V)([s]) = h^{\#}(s)$. This is well-defined because if $s \in \mathcal{O}_Y(V)$ is nilpontent, then $h^{\#}(s)$ must be nilpotent in $\mathcal{O}_X(h^{-1}(V))$, which means it must be equal to 0 as X is reduced. It follows that if s_1 and s_2 differ by a nilpotent element, then $h^{\#}(s_1) = h^{\#}(s_2)$. It is easy to see that this is a local morphism of presheaves from here, and is unique via the formula we have used to define it. This completes the proof.

D. Problem 2.2.4

We want to show that $\alpha : \text{Hom}_{\text{Sch}}(X, \text{Spec}(A)) \to \text{Hom}_{\text{Ring}}(A, \Gamma(X, \mathcal{O}_X))$ is a bijection. In the case that $X = \text{Spec}(R)$ is an *affine* scheme, this is quite simple. We replace $\Gamma(X, \mathcal{O}_X)$ with R, via the isomorphism, in the definition of α above. Note tha we have the map $\beta : \mathrm{Hom}_{\mathrm{Ring}}(A, R) \to \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}(R), \mathrm{Spec}(A))$ which takes $\phi : A \to R$ to the induced map (Proposition 2.3). It is proved in Proposition 2.3 that β is a left-inverse of α. To prove that it is also a right inverse, note that ϕ induces $(f, f^{\#})$ where $f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$, and we have the map $f^{\#}(\mathrm{Spec}(A)) : \Gamma(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)}) \to \Gamma(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ which takes s_a, the constant section taking on value $a \in A$ at every point, to $\phi \circ s_a \circ f$, which takes on $\phi(a)$ at every point. Thus, this section can be identified with ϕ under the isomorphisms identifying the global sections with A and R respectively. It follows that β is a right inverse, and α is a bijection in this case.

To prove that this result holds for general schemes, pick some cover $\{U_\alpha\}$ for X such that X is affine when restricted to each open set. We can use the α maps to induce a bijection

$$\widetilde{\alpha} : \prod_{\alpha \in J} \mathrm{Hom}_{\mathrm{Sch}}(U_\alpha, \mathrm{Spec}(A)) \to \prod_{\alpha \in J} \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(U_\alpha, \mathcal{O}_X)) \tag{6}$$

Of course, we have the natural restriction map on each affine open set, yielding

$$R : \mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(A)) \to \prod_{\alpha \in J} \mathrm{Hom}_{\mathrm{Sch}}(U_\alpha, \mathrm{Spec}(A)) \tag{7}$$

On the other hand, suppose we have a collection of morphisms of schemes $(f_\alpha, f_\alpha^{\#}) : U_\alpha \to \mathrm{Spec}(A)$ such that $f_\alpha = f_\beta$ on $U_\alpha \cap U_\beta$ and given $f_\alpha^{\#}(V) : \mathcal{O}(V) \to \mathcal{O}_X(f_\alpha^{-1}(V))$ and $f_\beta^{\#}(V) : \mathcal{O}(V) \to \mathcal{O}_X(f_\beta^{-1}(V))$, we have

$$f_\alpha^{\#}(V)(s)|_W = f_\beta^{\#}(V)(s)|_W \tag{8}$$

for $W \subset f_\alpha^{-1}(V) \cap f_\beta^{-1}(V)$. We immediately can combine the f_α to form a continuous map $f : X \to \mathrm{Spec}(A)$. Moreover, given some open $V \subset \mathrm{Spec}(A)$, and section $s \in \mathcal{O}(V)$, note that $f^{-1}(V)$ is covered by the U_α and $U_\alpha \cap f^{-1}(V) = f_\alpha^{-1}(V)$, so we can write $f^{-1}(V)$ as the union of the $f_\alpha^{-1}(V)$. Of course, the goal is to define a section on all of $f^{-1}(V)$. We have sections $f_\alpha^{\#}(V)(s) \in \mathcal{O}_X(f_\alpha^{-1}(V))$ for each α, and we know that these sections agree on intersections. Thus, by definition of a sheaf, they glue together to a unique section which restricts to the subsections.

It is easy to check that the subset of $\prod_\alpha \mathrm{Hom}_{\mathrm{Sch}}(U_\alpha, \mathrm{Spec}(A))$ which is the image of R is exactly the set with the restrictions described above, and that the map described above is an inverse to R. We use similar logic to treat the natural map

$$Q : \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X)) \to \prod_{\alpha \in J} \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(U_\alpha, \mathcal{O}_X)) \tag{9}$$

where we restrict global sections to local sections. In particular, given some element of the product determined by a collection of homomorphisms ϕ_α, we consider the case where sections $\phi_\alpha(a)$ and $\phi_\beta(a)$ agree on $U_\alpha \cap U_\beta$. It is clear that these elements are the image of Q. Moreover, when restricting to the image of R, we can easily see that these elements are the image of $\widetilde{\alpha}|_{\mathrm{Im}(R)}$. In particular, the maps $f_\alpha^{\#}(\mathrm{Spec}(A))$ and $f_\beta^{\#}(\mathrm{Spec}(A))$ which take $\Gamma(\mathrm{Spec}(A), \mathcal{O})$ to $\Gamma(U_\alpha, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X)$ respectively take global sections of $\mathrm{Spec}(A)$ to pairs which agree on $U_\alpha \cap U_\beta$.

Moreover, given some collection of ϕ_α which satisfy the desired criteria, they will induce an element of $\mathrm{Im}(R)$.

Thus, we have a commutative square where three of the arrows are bijections, so the fourth arrow between $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(A))$ and $\mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X))$ is also a bijection.

E. Problem 2.2.5

This follows quite readily from the previous solution: we have a bijection $\alpha : \mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(A)) \to \mathrm{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O}_X))$. Thus, our bijection is between $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(\mathbb{Z}))$ and $\mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}, \Gamma(X, \mathcal{O}_X))$. But of course, if $\phi : \mathbb{Z} \to R$ is an arbitrary ring homomorphism, then $\phi(1) = 1_R$, and this determines where every element of the domain is sent. Therefore, there is a unique element in $\mathrm{Hom}_{\mathrm{Sch}}(X, \mathrm{Spec}(\mathbb{Z}))$, as desired.

We are also asked to describe $\mathrm{Spec}(\mathbb{Z})$: it is the collection of prime ideals (p), each of which are closed points. The closed sets $V(k)$ can be broken down via a prime factorization of k to get a finite union of the single point sets $V(p_1) \cup \cdots \cup V(p_n)$, so along with the entire set $\mathrm{Spec}(\mathbb{Z})$, these finite sets give the topology (this is usually called the finite-complement topology).

F. Problem 2.2.7

A morphism is of the form $(f, f^{\#}) : \mathrm{Spec}(K) \to X$. We know that $\mathrm{Spec}(K)$ consists of a single point, (0), so specifying f is equivalent to specifying a point $x = f((0)) \in X$. The morphism of sheaves $f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_{\mathrm{Spec}(K)}$ is equivalent to specifying ring morphisms $f^{\#}(U) : \mathcal{O}_X(U) \to \mathcal{O}_{\mathrm{Spec}(K)}((0))$ for all U which contain the point x which are mutually compatible. Of course, the ring $\mathcal{O}_{\mathrm{Spec}(K)}((0))$ consists of all functions $s : \{(0)\} \to K_{(0)} \simeq K$, which is isomorphic to K itself. Hence, we may specify a ring morphism $\phi(U) : \mathcal{O}_X(U) \to K$ for each U containing x, such that if $V \subset U$ and ι is the inclusion, then $\phi(U) = \phi(V) \circ \mathcal{O}_X(\iota)$. Specifying all such morphisms is clearly equivalent to specifying a local ring morphism $\phi : \mathcal{O}_x \to K$. Finally, if ϕ is a local ring morphism, then $\phi^{-1}(0) = \mathfrak{m}_x$, so we obtain an injective morphism $\widetilde{\phi} : \mathcal{O}_x/\mathfrak{m}_x \to K$. Conversely, if $\widetilde{\phi}$ is a pre-specified injective morphism, then $\phi = \widetilde{\phi} \circ \pi$, where we pre-compose with the projection map, is a local ring homomorphism $\mathcal{O}_x \to K$, and it is easy to see that these correspondences are inverses of each other. We are then able to promote this homomorphism to a morphism of sheaves by defining $f^{\#}(U)$ to be 0 for U not containing x and localization at x followed by ϕ for all other U.

G. Problem 2.2.10

It is a basic fact from algebra that any (monic) polynomial in $\mathbb{R}[x]$ can be factored as a product of monomials $x - a$ and irreducible quadratics $x^2 + bx + c$. Thus, Since $\mathbb{R}$ is Noetherian, so is $\mathbb{R}[x]$. If $\mathfrak{p} = (f_1, \ldots, f_n)$ is a prime ideal, then it must contain each of the irreducible factors of each f_j of the above form, so any prime is generated by a finite collection of irreducibles of the form introduced in the first sentence. We cannot have $x - a$ and $x - b$ in this generating set for $a \neq b$, as then $1 \in \mathfrak{p}$ by taking their difference to obtain a unit. In addition, if $\mathfrak{p}$ contained both $x - a$ and $x^2 + bx + c$, then $\mathfrak{p}$ contains

$$x^2 + bx + c - (x - a)(x + b + a) = c + ab + a^2 \tag{10}$$

This must be equal to 0, otherwise $\mathfrak{p}$ would contain a unit, but then a would be a root of $x^2 + bx + c$, which can't be as this polynomial is irreducible. Thus, the prime ideals are precisely the principal ones $(x - a)$ and $(x^2 + bx + c)$ generated by the irreducible elements. Therefore, $\mathrm{Spec}(\mathbb{R}[x])$ has a point corresponnding to each $a \in \mathbb{R}$, namely $(x - a)$. It also has a point corresponding to each *conjugate pair* of complex numbers in $\mathbb{C}$. Thus, $\mathrm{Spec}(\mathbb{R}[x])$ has more points than $\mathbb{R}$, but less than $\mathbb{C}$.

If we want to compare topologies, note that of $\mathfrak{a} = (g_1, \ldots, g_m)$ is a polynomial ideal, we have $V(\mathfrak{a}) = \cap_{j=1}^m V(g_j)$. Each $V(g_j)$ is the union of $V(g_j^{(k)})$, where the $g_j^{(k)}$ are the irreducible factors. Each of these is a one-point set, so $V(g_j)$ is a finite union of points, and $V(\mathfrak{a})$ is also a finite union of points. Thus, the closed sets are precisely the finite collection of points, and the topology is thus the finite complement topology: quite different from the usual topologies on $\mathbb{R}$ and $\mathbb{C}$!

H. Problem 2.2.13

I. Problem 2.2.16

Part A. Let us first look at the affine case. In particular, when $X = \mathrm{Spec}(B)$, a choice of a global section amount to a choice of element f in B, the stalk $\mathcal{O}_{\mathfrak{p}}$ at point $\mathfrak{p}$ is isomorphic to $A_{\mathfrak{p}}$, and $\mathfrak{m}_{\mathfrak{p}} \simeq \phi(\mathfrak{p})$, the image of $\mathfrak{p}$ in the localization. The stalk of a section is then just the image of f in each of the localized rings $A_{\mathfrak{p}}$. The condition of $\rho(f)$ not being in $\phi(\mathfrak{p})$ is equivalent to the condition that $f \notin \rho^{-1}(\phi(\mathfrak{p})) = \mathfrak{p}$ (see the appendix). It follows that $\mathfrak{p} \in X_f$ if and only if $\mathfrak{p}$ does not contain (f) (equivalent to $\mathfrak{p} \in D((f))$). So, $X_f = D((f))$, in the affine case.

Now, for the general case, we can restrict to affine U and restrict a section f to $\overline{f}$, and note from above that $X_f \cap U = U_f = D((f))$.

Part B. Again, it is easy to prove this in the affine case. In this case, $X_f = D((f))$. In this case, restricting some $a \in A$, thought of as a global section, to $\mathcal{O}(D(f)) \simeq A_f$ amounts to mapping it under inclusion into the localization. a is 0 in the localization if and only if $f^n a = 0$ in A, for some n. To prove the general case, take a finite affine open cover for X by $U_1, \ldots, U_m$, and gets some n_k for each U_k so that $f^{n_k} a = 0$ in $\Gamma(U_k, \mathcal{O}_X|_{U_k})$. Taking the largest of the n_k to be n, and we get $f^n a = 0$ on all of the open sets U_k, thus on all of X.

Part C. Let us first look at the affines U_i. In particular, let us look at b restricted to $X_f \cap U_i \simeq D(f)$, so $b \in \mathcal{O}(D(f)) \simeq A_f$. It follows that $b = \frac{a_i}{f^{n_i}}$ for $a_i \in A$, on $U_i \cap X_f$. On the intersections $U_i \cap U_j \cap X_f = (U_i \cap U_j)_f$, we have course must have

$$\frac{a_i}{f^{n_i}} = \frac{a_j}{f^{n_j}} \Longleftrightarrow f^{m_{ij}}(a_i f^{n_j} - a_j f^{n_i}) = 0 \tag{11}$$

Then, since these intersections are quasi-compact, it follows from above that there exists some M_{ij} such that $f^{M_{ij}}(a_i f^{n_j} - a_j f^{n_i}) = 0$ on all of $U_i \cap U_j$. Let M be the largest of all the M_{ij}. We then have $f^M(a_i f^{n_j} - a_j f^{n_i})$ for every pair i and j. It follows that we just define $a = a_i f^{M+n_1+\cdots+\hat{n_i}+\cdots+n_K}$ for every i. Let $N = n_1 + \cdots + n_K$. Then, when we restrict to $U_i \cap X_f$, we have $a = f^{M+N} b$, for each i, and we are done.

Part D. Given some $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, we have that $f^n b$ is the restriction of some $a \in A$. Thus, we map b to $\frac{a}{f^n}$ in A_f. This is well-defined because if $f^m b$ is also a restriction of some $a' \in A$, we have

$$f^m a = f^{n+m} b = f^n a' \Longrightarrow \frac{a}{f^n} = \frac{a'}{f^m} \in A_f \tag{12}$$

It is clear that this map is also a ring homomorphism. If $f^n b$ is the restriction of 0, so $f^n b = 0$, then locally, where b can be thought of as an element of $\mathcal{O}(D(f)) \simeq A_f$, it follows that $b = 0$, so we have injectivity. Surjectivity follows from the fact that given some $\frac{a}{f^n}$, we can always choose b to be of this form locally in $\mathcal{O}(D(f))$, such a b will be mapped to $\frac{a}{f^n}$. Thus, we have the desired isomorphism.

III. Extra Proofs

Remark III.1 (Some details on localizations)**.** Let R be a ring, let S be a multiplicative set. Let $\rho : R \to S^{-1}R$ be the localization map taking r to $\frac{r}{1} \in S^{-1}R$. If $\mathfrak{a}$ is a (prime) ideal in $S^{-1}R$, then $\rho^{-1}(\mathfrak{a})$ is a (prime) ideal in R. We also define a map ϕ taking an ideal $\mathfrak{a}$ of R to the ideal in $S^{-1}R$ generated by the set $\rho(\mathfrak{a})$. It is easy to see that every element of $S^{-1}\mathfrak{a}$ is of the form $\frac{r}{s}$, for some $r \in \mathfrak{a}$. Let us consider the relationship between ϕ and the map ρ^{-1} taking $\mathfrak{a}$ to $\rho^{-1}(\mathfrak{a})$. Our first claim is that $\mathfrak{a} = \phi(\rho^{-1}(\mathfrak{a}))$ for *any* ideal $\mathfrak{a}$. To see this, pick some $\frac{r}{s} \in \mathfrak{a} \subset S^{-1}R$. Note that $\rho(r) = \frac{r}{1} = \frac{sr}{s} \in \mathfrak{a}$, so $r \in \rho^{-1}(\mathfrak{a}) \subset R$ and $\frac{r}{1} \in \rho(\rho^{-1}(\mathfrak{a}))$. This means that $\frac{r}{s}$ is in the ideal in $S^{-1}R$ generated by all images of elements of $\rho^{-1}(\mathfrak{a})$ under ρ, so $\mathfrak{a} \subset \phi(\rho^{-1}(\mathfrak{a}))$. On the other hand, given some element of $\phi(\rho^{-1}(\mathfrak{a}))$, it will be of the form

$$\frac{r_1}{s_1} \cdot \frac{q_1}{1} + \cdots + \frac{r_m}{s_m} \cdot \frac{q_m}{1} = \frac{r_1' q_1 + \cdots + r_m' q_m}{s_1 \cdots s_m} \tag{13}$$

so any such element is of the form $\frac{r}{s}$ for some $r \in \rho^{-1}(\mathfrak{a})$. Thus, $\frac{r}{1} \in \mathfrak{a}$, so $\frac{r}{s} \in \mathfrak{a}$, which gives the reverse inclusion.

Thus, $(\phi \circ \rho^{-1})(\mathfrak{a}) = \mathfrak{a}$. However, it is *not* the case that $(\rho^{-1} \circ \phi)(\mathfrak{a}) = \mathfrak{a}$. This only works if we restrict ϕ to the collection of prime ideals which do not intersect S, and ρ^{-1} to the collection of prime ideals in $S^{-1}R$. Note that if $\mathfrak{p} \subset S^{-1}R$ is prime, then $\rho^{-1}(\mathfrak{p})$ will be prime in R. Moreover, if $s \in \rho^{-1}(\mathfrak{p})$ for some $s \in S$, then $\frac{s}{1} \in \mathfrak{p}$, so $\frac{1}{1} \in \mathfrak{p}$, contradicting the fact that it must be proper. On the other hand, if $\mathfrak{p}$ is a prime ideal in R not intersecting S, and if some $\frac{p}{s} \in \phi(\mathfrak{p})$ with $p \in \mathfrak{p}$, $s \in S$ can be written as $\frac{p}{s} = \frac{p_1}{s_1} \cdot \frac{p_2}{s_2} = \frac{p_1 p_2}{s_1 s_2}$, then $q s_1 s_2 p = q s p_1 p_2$ for some $q \in S$, so $q s p_1 p_2 \in \mathfrak{p}$. Since $S \cap \mathfrak{p} = \emptyset$ and $\mathfrak{p}$ is prime, $p_1 p_2 \in \mathfrak{p}$, so either p_1 or p_2 is. Then either $\frac{p_1}{s_1}$ or $\frac{p_2}{s_2}$ is in $\phi(\mathfrak{p})$.

Let us finally show that $(\rho^{-1} \circ \phi)(\mathfrak{p}) = \mathfrak{p}$ for such prime ideals. First note that given $p \in \mathfrak{p}$, then $\rho(p) = \frac{p}{1}$ is in $\phi(\mathfrak{p})$, so $\mathfrak{p} \subset (\rho^{-1} \circ \phi)(\mathfrak{p})$. Note that we haven't yet used the fact that $\mathfrak{p}$ is prime: this inclusion holds for any ideal. However, we need this assumption for the reverse inclusion. If $p \in (\rho^{-1} \circ \phi)(\mathfrak{p})$, then $\frac{p}{1} \in \phi(\mathfrak{p})$. Thus, it can be written as $\frac{q}{s}$ for some $q \in \mathfrak{p}$ and $s \in S$. Then we have $t(sp - q) = 0$ for some $t \in S$, so $sp \in \mathfrak{p}$. Then, since $\mathfrak{p}$ is prime and does not contain any elements of S, $p \in \mathfrak{p}$, giving the reverse inclusion.

The above discussion gives us the following result:

Lemma III.1. If R is a ring, and S is a multiplicative set, then the prime ideals of $S^{-1}R$ are in bijective correspondence with the prime ideals of R which do not intersect S

Let us continue our discussion of localizations.

Lemma III.2. If S and T are multiplicative sets in ring R such that $S \subset T$, then $S^{-1}T$ is a multiplicative set and $T^{-1}R \simeq (S^{-1}T)^{-1}(S^{-1}R)$ as rings.

Proof. Define the map $\Phi : T^{-1}R \to (S^{-1}T)^{-1}(S^{-1}R)$ to be $\Phi(\frac{r}{t}) = \frac{r/1}{t/1}$. To show that this is well-defined, suppose $\frac{r'}{t'} = \frac{r}{t}$. Then $q(tr' - t'r) = 0$ for some $q \in T$. We want to show that $\frac{r'/1}{t'/1} = \frac{r/1}{t/1}$, which is true if and only if there is $\frac{x}{s} \in S^{-1}T$ such that

$$\frac{x}{s}\left(\frac{t'r}{1} - \frac{tr'}{1}\right) = \frac{x(t'r - tr')}{s} = 0 \tag{14}$$

so we just let $\frac{x}{s} = \frac{q}{1}$, and we have what we want. The fact that this is a homomorphism is easy. Given some $\frac{r/s_1}{t/s_2}$, note that $\Phi(\frac{s_2 r}{s_1 t}) = \frac{s_2 r/1}{s_1 t/1}$. We claim this is equal to the first expression. Indeed,

$$\frac{s_1 t}{1}\frac{r}{s_1} - \frac{s_2 r}{1}\frac{t}{s_2} = \frac{r}{t} - \frac{r}{t} = 0 \tag{15}$$

so this is the case, and we have surjectivity. Finally, suppose $\Phi(\frac{r}{t}) = \frac{r/1}{t/1} = 0$. Then $\frac{x}{s}\frac{r}{1} = \frac{xr}{s} = 0$ for some $\frac{x}{s} \in S^{-1}T$. Then $x'r = 0$ for some $x' \in T$. It follows that $\frac{r}{t} = \frac{x'r}{x't} = 0$ in $T^{-1}R$, so we have injectivity. Thus, Φ is an isomorphism. ☐

Corollary III.0.1. It follows that if A is a ring, and $\mathfrak{p}$ is a prime ideal which does not contain f, so $S = \{1, f, f^2, \dots\}$ is contained in $T = A - \mathfrak{p}$, then

$$A_{\mathfrak{p}} = T^{-1}A \simeq (S^{-1}T)^{-1}(S^{-1}A) = (S^{-1}(A - \mathfrak{p}))^{-1}(S^{-1}A) = (A_f - S^{-1}\mathfrak{p})^{-1}A_f = (A_f)_{S^{-1}\mathfrak{p}} \tag{16}$$

This will be useful in some of the problems.

Here is a fact about sheaves:

Proposition III.1. If $f : X \to Y$ is a homeomorphism and $\mathcal{O}_X$ is a sheaf on X, then $(f_*\mathcal{O}_X)_{f(x)}$, the stalk of the pushforward sheaf, is isomorphic to $\mathcal{O}_{X,x}$.

Proof. Note that the elements of $\mathcal{O}_{X,x}$ are germs $[U, s]$ where $s \in \mathcal{O}_X(U)$ and $x \in U$. The elements of $(f_*\mathcal{O}_X)_{f(x)}$ are germs $[V, r]$ with $r \in \mathcal{O}_X(f^{-1}(V))$ and $f(x) \in V$. Thus, we have map $f^\# : \mathcal{O}_{X,x} \to (f_*\mathcal{O}_X)_{f(x)}$ with $f^\#[U, s] = [f(U), s]$. This is obviously a ring homomorphism, and it has an inverse. ☐