SPIVAK NOTES, PROBLEMS, AND SOLUTIONS

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CONTENTS

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1. Introduction

The goal of this set of notes is to solve the most challenging problems in Spivak, and write up the solutions in a clean and concise way. I apologize in advance for any possible mistakes, or instances in which I may skip over certain important points.

2. Chapter 3

Problem 3.17. Prove that if $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, where $f(x) \neq 0$, then $f(x) = x$ for all x.

Proof. We go through the steps of the proof, as organized in Spivak:

(1) Clearly, we will have $f(1) = f(1 \cdot 1) = f(1) \cdot f(1)$, so either $f(1) = 0$ or $f(1) = 1$. If we assume that $f(1) = 0$, then this would imply that $f(n) = 0$ for all n (we can prove this by induction, assuming that $f(n) = 0$, and noting that $f(n + 1) = f(n) + f(1) = 0$). This is a contradiction to our initial assumption, so $f(1) = 1$.

(2) First, we note that:

$$f(0) = f(0 + 0) = f(0) + f(0) \ \Rightarrow \ f(0) = 0$$

Next, we note that $f(n) = n$, for natural n. We prove this by induction, first assuming that $f(n) = n$, then noting that $f(n + 1) = f(n) + f(1) = n + 1$. We then note that $f(-n) = n - n + f(-n) = -n + f(n) + f(-n) = -n + f(0) = -n$. Thus, f is the identity for all integers.

Now, we can see that:

$$f\left(\frac{1}{b}\right) = \frac{b}{b} \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(b) \cdot f\left(\frac{1}{b}\right) = \frac{1}{b} \cdot f(1) = \frac{1}{b}$$

Thus,

$$f\left(\frac{a}{b}\right) = f(a) \cdot f\left(\frac{1}{b}\right) = \frac{a}{b}$$

(3) Assume that $x > 0$. It then follows that $\sqrt{x}$ is well-defined and greater than 0. We then have:

$$f(x) = f(\sqrt{x} \cdot \sqrt{x}) = f(\sqrt{x})f(\sqrt{x}) = f(\sqrt{x})^2$$

we know that for any real number r, we have $r^2 \geq 0$, so $f(x) \geq 0$. Assume that $f(x) = 0$. Since $x > 0$, this would imply that:

$$f(1) = f\left(\frac{x}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right) = 0$$

a clear contradiction. Thus, $f(x) > 0$.

(4) If $x > y$, then we know that $x - y > 0$, so it follows from previous result that:

$$f(x - y) > 0 \ \Rightarrow \ f(x) - f(y) > 0 \ \Rightarrow \ f(x) > f(y)$$

(5) Assume that there exists some x such that $x < f(x)$. Since there exists a rational number between any two reals, it follows that we have:

$$x < \frac{a}{b} < f(x)$$

for some a/b. From the previous result, we then get $f(x) < f(a/b)$, a clear contradiction to the right-most inequality above. Similarly, if we assume that $f(x) < x$, we will have:

$$f(x) < \frac{a}{b} < x$$

so $f(a/b) < f(x)$, another contradiction. It follows that $f(x) = x$, and we have proved the proposition.

$\square$

Problem 3.20B. If a function satisfies:

$$f(y) - f(x) \leq (x - y)^2$$

for all $, y \in \mathbb{R}$, then $f(x) = c$ for some c and all x

Part B is the interesting part of this problem, so I skipped writing out Part A

Proof. Assume that there exist distinct x and y such that $f(x) \neq f(y)$. Without loss of generality, let $f(x) < f(y)$. It follows that:

$$f(y) - f(x) \leq (y - x)^2$$

Consider what happens when we split up the interval from x to y into n "chunks". We let:

$$z_j = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y$$

so we get $z_0 = x$ and $z_n = y$. Clearly the distance between z_i and z_{j-1} is given by:

$$z_j - z_{j-1} = \left(1 - \frac{j}{n}\right)x + \frac{j}{n}y - \left(1 - \frac{j-1}{n}\right)x - \frac{j-1}{n}y = \frac{y - x}{n}$$

It follows that:

$$f(z_j) - f(z_{j-1}) \leq (z_j - z_{j-1})^2 = \frac{(y - x)^2}{n^2}$$

Now comes the crucial step. Notice that

$$\sum_{j=1}^{n} \left(f(z_j) - f(z_{j-1})\right) = f(z_n) - f(z_0) = f(y) - f(x)$$

as the rest of the terms cancel. Thus, we will have:

$$\sum_{j=1}^{n} \left(z_j - z_{j-1}\right) \leq \sum_{j=1}^{n} \frac{(y - x)^2}{n^2} \implies f(y) - f(x) \leq n \cdot \frac{(y - x)^2}{n^2} = \frac{(y - x)^2}{n}$$

for all possible values of n. Since we have assume $f(y) \neq f(x)$, it follows that $f(y) - f(x)$ is a positive real number, and that:

$$\epsilon = \frac{f(y) - f(x)}{(y - x)^2}$$

is a positive real as well. Thus, this implies that there exists a real number ϵ such that for any positive integer n:

$$\epsilon \leq \frac{1}{n}$$

But this clearly contradicts the Archimedean property of the real numbers. We have derived a contradiction, so it follows that for any x and y, $f(x) = f(y)$. Thus, the function f is constant.

$\square$

3. Chpater 5

Lemma 1 (Uniqueness of Limits). *The limit of a function is unique: If a function f approaches ℓ_1 as x approaches a, and f approaches ℓ_2 as x approaches a, then $\ell_1 = \ell_2$.*

Proof. Suppose the the function f approaches ℓ_1 and ℓ_2. It follows that given some $\epsilon > 0$, we can choose δ_1 and δ_2 such that:

$$|x - a| < \delta_1 \implies |f(x) - \ell_1| < \epsilon$$
$$|x - a| < \delta_2 \implies |f(x) - \ell_2| < \epsilon$$

Assume that $\ell_1 \neq \ell_2$, so $|\ell_1 - \ell_2| > 0$. Let us then pick $\epsilon = \frac{|\ell_1 - \ell_2|}{2}$. We can then pick δ_1 and δ_2 corresponding to this ϵ. We then let $\delta = \min(\delta_1, \delta_2)$ so:

$$|x - a| < \delta \implies |f(x) - \ell_1| < \epsilon \quad \text{and} \quad |f(x) - \ell_2| < \epsilon$$

It then follows that:

$$|x - a| < \delta \implies |f(x) - \ell_1| + |f(x) - \ell_2| < 2\epsilon = |\ell_1 - \ell_2|$$

We know that there exists some x_0 such that $|x_0 - a| < \delta$, which implies that:

$$|\ell_1 - \ell_2| \leq |f(x_0) - \ell_1| + |f(x_0) - \ell_2| < |\ell_1 - \ell_2|$$

a clear contradiction. It follows that ℓ_1 must equal ℓ_2.

$\square$

Lemma 2 (Sums of Limits)**.** *If* $\lim_{x \to a} f(x) = m$ *and* $\lim_{x \to a} g(x) = \ell$, *then* $\lim_{x \to a}(f + g)(x) = m + \ell$.

Proof. Let us pick some $\epsilon > 0$. We will have, for $\epsilon/2$:

$$|x - a| < \delta_1 \implies |f(x) - m| < \epsilon/2$$
$$|x - a| < \delta_2 \implies |g(x) - \ell| < \epsilon/2$$

we choose $\delta = \min(\delta_1, \delta_2)$, giving us:

$$|x - a| < \delta \implies |f(x) - m| + |g(x) - \ell| < \epsilon$$

Then, given x such that $|x - a| < \delta$, we have:

$$|f(x) + g(x) - (m + \ell)| \leq |f(x) - m| + |g(x) - \ell| < \epsilon$$

Thus, given ϵ, we can choose a δ. It follows by definition that $\lim_{x \to a}(f + g)(x) = m + \ell$.

$\square$

Problem 5.20. If $f(x) = x$ for rational x and $f(x) = -x$ for irrational x, show that $\lim_{x \to a} f(x)$ does not exist for $a \neq 0$.

Proof. Assume that there exists some non-zero a such that:

$$\lim_{x \to a} f(x) = L$$

It follows that for any $\epsilon > 0$, we can choose a δ such that if $|x - a| < \delta$, then $|f(x) - L| < \epsilon$. We begin by considering the case when $a > 0$. We let $\epsilon = a$ and assume that we can choose a δ such that:

$$|x - a| < \delta \implies |f(x) - L| < a$$

Now, since there exists a rational and an irrational number between any two reals, we pick rational r and irrational i from the interval $(a, a + \delta)$. We will then have:

$$|r - a| < \delta \implies |r - L| < a$$
$$|i - a| < \delta \implies |-i - L| = |i + L| < a$$

So we will have:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2a$$

But this is a contradiction, as $a < i, r$, so $2a < i + r$. Thus, we can choose no such $\delta > 0$, and the limit does not exist.

Now, assume that $a < 0$. We let $\epsilon = |a|$ and assume that we can choose a corresponding δ. We then choose rational and irrational $r, i \in (a - \delta, a)$. Similar to above, we get:

$$|(r - L) + (i + L)| = |r + i| \leq |r - L| + |i + L| < 2|a|$$

But this is a contradiction, as $i, r < a$, so $i + r < 2a$, which implies that $|i + r| > 2|a|$ (as a, i, and r are negative). Thus, we can choose no such δ, and the limit does not exist.

We conclude that the limit does not exist for any $a > 0$, and any $a < 0$, making $a = 0$ the only point at which the limit exists. $\qquad\square$

Problem 5.12. Suppose that $f(x) \leq g(x)$ for all x. Prove that $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$, assuming the limits exist.

Proof. We let the first limit be denoted by ℓ_f and the second by ℓ_g. Assume that $\ell_f > \ell_g$. It follows that $\ell_f - \ell_g$ is a positive real number, so we let $\epsilon = (\ell_f - \ell_g)/2$. Now, by definition of the limits, we can choose δ_1 and δ_2 such that:

$$0 < |x - a| < \delta_1 \;\Rightarrow\; |f(x) - \ell_f| < (\ell_f - \ell_g)/2$$
$$0 < |x - a| < \delta_2 \;\Rightarrow\; |g(x) - \ell_g| < (\ell_f - \ell_g)/2$$

.
We let $\delta = \min\{\delta_1, \; \delta_2\}$. We then have:

$$0 < |x - a| < \delta \;\Rightarrow\; |\ell_f - f(x) + g(x) - \ell_g| \leq |f(x) - \ell_f| + |g(x) - \ell_g| < \ell_f - \ell_g$$

So we have:

$$|(g(x) - f(x)) + (\ell_f - \ell_g)| < \ell_f - \ell_g$$

but since $f(x) \leq g(x)$ and $\ell_g < \ell_f$, both numbers in the brackets will be greater than or equal to 0, so:

$$g(x) - f(x) + \ell_f - \ell_g < \ell_f - \ell_g$$

which is a contradiction. Thus, $\ell_f \leq \ell_g$ and the proof is complete.

$\qquad\square$

Problem 5.23. Let f be a function with the following property: if g is a function for which $\lim_{x \to 0} g(x)$ does not exists, then $\lim_{x \to 0}[f(x) \cdot g(x)]$ also does not exist. Prove that f has this property if and only if $\lim_{x \to 0} f(x)$ exists.

Proof. We start by considering the case where $\lim_{x \to 0} f(x)$ exists and is equal to $m \neq 0$. Let $g(x)$ be a function such that $\lim_{x \to 0} g(x)$ does not exist. Assume that $\lim_{x \to 0}[f(x) \cdot g(x)]$ exists, so it is equal to some real ℓ. We then have:

$$\frac{\lim_{x \to 0}[f(x) \cdot g(x)]}{\lim_{x \to 0} f(x)} = \lim_{x \to 0} \left(\frac{f(x) \cdot g(x)}{f(x)} \right.$$

$\qquad\square$

Problem 5.24. Suppose that A_n is, for each natural n, some finite set of numbers of $[0, \; 1]$, and that A_n and A_m are disjoint if $n \neq m$. Define f as follows:

$$f = \begin{cases} 1/n & x \in A_n \\ 0 & x \notin A_n \forall n \in \mathbb{N} \end{cases}$$

Prove that $\lim_{x \to a} f(x) = 0$ for any $a \in [0, \; 1]$.

Proof. Let us pick some $a \in [0, \; 1]$ and some $\epsilon > 0$. By the Archimedean property, we can pick some natural n such that $1/n < \epsilon$. Since each A_m contains only a finite number of elements, it follows that the union of the collection of set $\{A_1, \; ..., \; A_{n-1}\}$ also contains a finite number of elements.

By definition of f, this implies that there are a finite number of $x \in [0, \; 1]$ such that $1/n < f(x)$. We denote the set of such x by X. Then, we let:

$$\delta = \min\{|x - a| \mid x \in X - \{a\}\}$$

where the minimum of the set is well-defined, as X contains a finite number of elements. It then follows that given some y such that $0 < |y - a| < \delta$, y cannot possibly be in X, so it must be true that $f(x) \leq 1/n < \epsilon$.

$\square$

4. CHAPTER 7

Problem 7.5. Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f?

Proof. We can say that f is constant, $f(x) = c$.

Let us pick x and y with $x \neq y$. Assume that $f(x) \neq f(y)$. Without loss of generality, we say $f(x) < f(y)$. Both $f(x)$ and $f(y)$ are rational. We know that there will be in irrational number r in the interval $[f(x), f(y)]$. But since f is continuous, by IVT, there must be some z with $f(z) = r$, a contradiction. Thus, $f(x)$ must equal $f(y)$, so f is constant. $\square$

Problem 7. Suppose that ϕ is continuous and $\lim_{x \to \infty} \phi(x)/x^n = 0 = \lim_{x \to -\infty} \phi(x)/x^n$.

Prove that if n is odd, then there is a number x such that $x^n + \phi(x) = 0$

Proof. Since ϕ and x^n are both continuous, the function $g(x) = x^n + \phi(x)$ is also continuous. We also note that:

$$\lim_{x \to \pm\infty} \frac{g(x)}{x^n} = \lim_{x \to \pm\infty} \left(1 + \frac{\phi(x)}{x^n}\right) = 1$$

So it follows that:

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \frac{g(x)}{x^n} x^n = \lim_{x \to \infty} x^n \lim_{x \to \infty} \frac{g(x)}{x^n} = \lim_{x \to \infty} x^n = \infty$$

and similarly:

$$\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} \frac{g(x)}{x^n} x^n = -\infty$$

Thus, it follows that there must exist C such that $f(C) < 0$ and D such that $f(D) > 0$. Thus, we apply intermediate value theorem to the interval $[C, D]$, to conclude there must exist some x such that $g(x) = x^n + \phi(x) = 0$. This completes the proof. $\square$

Prove that if n is even, then there is a number y such that $y^n + \phi(y) \leq x^n + \phi(x)$ for all x.

Proof. Again, we let $g(x) = x^n + \phi(x)$ and note that the limit as $g(x)/x^n$ approaches $\pm\infty$ is 1. From this, we proceed in a similar fashion to the previous proof, noting that the limit as $g(x)$ approaches $\pm\infty$ is ∞.

Thus, we can choose some A such that $x > A > 0 \Rightarrow g(x) > g(0)$, and some B such that $x < B < 0 \Rightarrow g(x) > g(0)$. We then apply the extreme value theorem to the interval $[A, B]$ to conclude that this interval has a minimum, which occurs at the point $x = y$. We note that all points not in this interval are greater than $g(0)$, which is itself in the interval, so y must be the global minimum of the function. By definition:

$$y^n + \phi(y) \leq x^n + \phi(x)$$

for all x and the proof is complete. $\square$

Problem 7.18. Suppose the f is continuous on $[a, b]$ and let x be any number. Prove that there is a point on the graph of f that is closest to $(x, 0)$.

Proof. We note that $\square$

5. CHAPTER 8

Problem 8.20a. Let (a, b) be an interval of shadow points, with a and b not shadow points. Prove that for all $x \in (a, b)$, we have $f(x) \leq f(b)$.

Proof. Consider some point $x \in (a, b)$. We let $A = \{y : x \leq y \leq b \text{ and } f(x) \leq f(y)\}$. Assume that $\sup A < b$. It follows that $\sup A \in (a, b)$, so $\sup A$ is a shadow point.

It follows that there must exist some $y > \sup A$ such that $f(y) > f(\sup A)$. In addition, since $b \notin A$, we must have $f(b) < f(x)$. If $y \in [x, b]$, with $f(y) > f(\sup A) \geq f(x)$, then y would be an element of A greater than $\sup A$, a contradiction. Thus, $b < y$. Thus, we have found a point y such that $b < y$ and $f(b) < f(y)$. Thus, b is a shadow point, a contradiction.

Since b is an upper-bound on A, it follows that $\sup A = b$, for any x. □

Problem 8.20b. Prove that $f(a) \leq f(b)$.

Proof. Assume that $f(a) > f(b)$. It follows that for any c with $f(b) < c < f(a)$, we can choose $r \in (a, b)$ such that $f(r) = c$ (by IVT). But this is a clear contradiction to Part A. Thus, $f(a) \leq f(b)$. □

Problem 8.20c. Prove that $f(a) = f(b)$.

Proof. Assume that $f(a) < f(b)$. It would follow from the definition that a is a shadow point, a contradiction. Therefore, $f(a) = f(b)$. □

6. CHAPTER 9

Problem 9.15. let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at x.

Proof. It is clear that for all x, we have $0 \leq |f(x)|$. Thus:

$$0 \leq |f(x)| \leq x^2 \;\Rightarrow\; 0 \leq \frac{|f(x)|}{x} \leq x$$

for all $x \neq 0$. We then invoke squeeze theorem to conclude that:

$$f'(0) = \lim_{x \to 0} \frac{|f(x)|}{x} = 0$$

and the proof is complete. □

Generalize this result be replacing x^2 with $|g(x)|$. What property must g have?

Proposition 1. *If $g(0) = 0$, then the derivative of f at 0 exists.*

Proof. Similarly to above, we note that $0 \leq |f(x)| \leq |g(x)|$ so $0 \leq |f(x)|/x \leq |g(x)|/x$ for all $x \neq 0$. Then, by squeeze theorem, $f'(0) = 0$ □

7. CHAPTER 10

Problem 10.19a. Suppose that f is continuous on $[0, 1]$ and $f(0) = f(1)$. Let n be any natural number. Prove that there is some number x such that $f(x) = f(x + 1/n)$.

Proof. Consider the continuous function $g(x) = f(x) - f(x + 1/n)$. Suppose that $g(x) \neq 0$ for all x. The intermediate value theorem implies that $g(x)$ must then either be positive for all x, or negative for all x, as if it took on both positive and negative values, it would have to take on the value 0.

This would then imply that $f(x) > f(x + 1/n)$ or $f(x) < f(x + 1/n)$ for all x. In either case, we would have $f(0) > f(1/n)$ or $f(1/n) < f(0)$. It is then easy to prove inductively that for all k from 1 to n, we will have $f(0) > f(k/n)$ or $f(k/n) < f(0)$, which gives us $f(0) > f(1)$ or $f(1) < f(0)$, both clear contradictions.

Thus, there must exist a value of x for which $g(x) = 0$, implying that $f(x) = f(x + 1/n)$. □

Problem 10.19b.

Problem 10.20a. Prove that there does not exist a continuous function on $\mathbb{R}$ that takes on every value exactly twice.

Proof. Suppose that we have a function f such that it takes on every value exactly twice. It follows that given some value a, there exists exactly one other value b such that $f(a) = f(b)$. Assume without loss of generality that $a < b$.

It follows that we can apply the intermediate value theorem to the interval $[a, \ b]$, to conlude that either $f(a) < f(x)$ for all $x \in (a, \ b)$ or $f(x) < f(a)$ for all $x \in (a, \ b)$, as if the function changes sign, then there must be some $c \in (a, \ b)$ such that $f(a) = f(c) = f(b)$, a contradiction.

We can again assume without loss of generality that $f(x) < f(a)$ for all $x \in (a, \ b)$. Now, by extreme value theorem, there is a value $y \in (a, \ b)$ such that $f(y) \geq f(x)$ for all $x \in [a, \ b]$. Thus, we apply intermediate value theorem to the intervals $[a, \ y]$ and $[y, \ b]$ to find that every $c \in [f(a), \ f(y)]$, except $f(y)$, is equal to $f(x)$ for exactly two $x \in [a, \ b]$.

Since f takes on the same value exactly twice, it follows that for $x < a$ or $x > b$, we must have $f(x) < f(a)$, or else there would exist some value that the function takes on at least three times. But $f(a) \leq f(y)$, so it follows that $f(y)$ is only taken on by one value of x. This is a contradiction, so such a function cannot exist. $\square$

Problem 10.25a. Suppose that a and b are consecutive roots of $f(x)$, so $f(x) = (x-a)(x-b)g(x)$, where $g(a) \neq 0$ and $g(b) \neq 0$. Prove that $g(a)$ and $g(b)$ have the same sign.

Proof. Assume without loss of generality that $g(a)$ is positive, and that $g(b)$ is negative. It follows from the intermediate value theorem that there must exist some x such that $g(x) = 0$ and $a < x < b$, clearly contradicting the fact that a and b are consecutive roots. Thus, $g(a)$ and $g(b)$ must have the same sign. $\square$

Problem 10.25b. Prove that there is some x with $a < x < b$ and $f'(x) = 0$.

Proof. We note that:

$$f'(x) = (x - b)g(x) + (x - a)g(x) + (x - a)(x - b)g'(x)$$

Therefore, $f'(a) = (a - b)g(a)$. We can assume that $a < b$, so $a - b$ is negative. In addition, we note that $f'(b) = (b - a)g(b)$, with $b - a$ positive. Thus, $f'(a)$ and $f'(b)$ have opposite signs (as $g(a)$ and $g(b)$ have the same sign). It follows from IVT that there exists some point x between a and b such that $f'(x) = 0$. $\square$

Problem 10.25c. Prove the same fact, even if a and b are multiple roots.

Proof. We may write $f(x) = (x - a)^m(x - b)^n g(x)$. We note that:

$$f'(x) = m(x - a)^{m-1}(x - b)^n g(x) + n(x - a)^m(x - b)^{n-1}g(x) + (x - a)^m(x - b)^n g'(x)$$

We then define:

$$h(x) = \frac{f'(x)}{(x - a)^{m-1}(x - b)^{m-1}} = m(x - b)g(x) + n(x - a)g(x) + (x - a)(x - b)g'(x)$$

We use the same logic as in 10.25a to note that $g(a)$ and $g(b)$ must have the same sign. It then follows (suing the same logic as 10.25b) that there is some point x between a and b such that $h(x) = 0$. By definition, this implies that $f'(x) = 0$ and the proof is complete. $\square$

Problem 10.27. Suppose f is differentiable at 0, and that $f(0) = 0$. Prove that $f(x) = xg(x)$ for some function g which is continuous at 0.

Proof. Since f is differentiable at 0, it follow that:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}$$

is a well-defined real number. Thus, we define $g(x) = f(x)/x$ for all x other than 0 and $f'(0)$ when $x = 0$. Clearly, since f is differentiable, it is continuous, so:

$$\lim_{x \to a} g(x) = \lim_{x \to a} \frac{f(x)}{x} = \frac{f(a)}{a}$$

for non-zero a, and $f'(0)$ for $a = 0$. Thus, the function g is continuous at all a, and $f(x) = xg(x)$ for all x. This completes the proof. $\qquad\square$

Problem 10.28. Prove that it is impossible to write $x = f(x)g(x)$ where f and g are differentiable and $f(0) = g(0) = 0$

Proof. Assume that this is possible. It would follow from differentiating that:

$$f'(x)g(x) + g'(x)f(x) = 1$$

where $f'(x)$ and $g'(x)$ are well defined. We then must have $f'(0)g(0) + g'(0)f(0) = 1$, which is impossible, as $f(0) = g(0) = 0$. Thus, we cannot find f and g such that $f(x)g(x) = x$ and the proof is complete. $\qquad\square$

8. Chapter 11

Proposition 2. *If $f'(x) \geq M$, then $f(x) \geq Mx + f(0)$ for all $x \in [0, 1]$.*

Proof. Assume that there exsits a point y such that $f(y) < My + f(0)$. It follows from mean value theorem that there exists some z such that:

$$f'(z) = \frac{f(y) - f(0)}{y} < \frac{My + f(0) - f(0)}{y} = M$$

a clear contradiction. $\qquad\square$

Problem 11.26. Suppose that $f'(x) \geq M > 0$ for all $x \in [0, 1]$. Prove that there exists an interval of length $1/4$ such that $|f| \geq M/4$.

Proof. First, assume that $f(0) \geq -M/2$. Using the above proposition, we note that:

$$f(3/4) \geq \frac{3M}{4} + f(0) \geq \frac{3M}{4} - \frac{M}{2} = \frac{M}{4}$$

Then, since f is strictly increasing, we will have $|f| \geq M/4$ on the interval $[3/4, 1]$.

Now, consider the case where $f(0) < -M/2$. $\qquad\square$

Problem 11.27. Show that if $f'(x) > g'(x)$ for all x and $f(a) = g(a)$, then $f(x) > g(x)$ for all $x > a$ and $f(x) < g(x)$ for all $a < x$.

Proof. We make use of the Cauchy mean value theorem. Choose b such that $b > a$. It follows that we can choose some point x such that:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

But since $f'(x) > g'(x)$, we note that $f'(x)/g'(x) > 1$, so:

$$f(b) - f(a) > g(b) - g(a) \implies f(b) > g(b)$$

where we used the fact that $f(a) = g(a)$. Similar logic shows the case of $b < a$. $\qquad\square$

Problem 11.33. Suppose that $|f(x) - f(y)| \leq |x - y|^n$ for some $n > 1$. Prove that $f(x) = c$ for all x.

Proof. Consider $f'(x)$. We assert that $f'(a)$, for some a, is equal to 0. We pick some $\epsilon > 0$. We let $\delta = \min\{1, \ \epsilon\}$. We then have:

$$|x - a| < \delta \implies \left| \frac{f(x) - f(a)}{x - a} \right| = \frac{|f(x) - f(a)|}{|x - a|} \leq \frac{|x - a|^n}{|x - a|} = |x - a|^{n-1} \leq |x - a| < \delta \leq \epsilon$$

as $n > 1$ and $\delta \leq 1$. Thus, by definition, the derivative of f at a is equal to 0. It follows that $f(x) = c$ (this can be formally demonstrated using mean value theorem). $\qquad\square$

Problem 11.34a. If f is Lipschtiz of order $\alpha > 0$, at x, then f is continuous at x.

Proof. We □

Problem 11.39. Prove that if f is twice-differentiable function with $f(0) = 0$ and $f(1) = 1$, as well as $f'(0) = f'(1) = 0$, then $|f''(x)| \geq 4$ for some $x \in [0, 1]$.

Proof. **To do** □

Problem 11.40. Consider some function f such that $f'(x) = 1/x$ for all $x > 0$ and $f(1) = 0$. Prove that $f(xy) = f(x) + f(y)$ for all $x, y > 0$.

Proof. Let us consider $g(x) = f(xy)$, taking the derivative with respect to x:

$$\frac{dg}{dx} = yf'(xy) = y\frac{1}{xy} = \frac{1}{x} = \frac{df}{dx}$$

for all $x > 0$. We know that the derivatives of the functions f and g are the same, so it follows that $g(x) = f(xy) = f(x) + c$, for some constant c (by mean value theorem). We then note that for $x = 1$, we have:

$$f(y) = f(1) + c = c$$

Therefore, $f(xy) = f(x) + f(y)$ for all $x, y > 0$. □

Problem 11.47. Prove that if f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for x in (a, b), then there is some x in (a, b) with:

$$\frac{f'(x)}{g'(x)} = \frac{f(x) - f(a)}{g(b) - g(x)}$$

Proof. We let:

$$h(x) = g(x)f(x) - f(a)g(x) - g(b)f(x)$$

We note that $h(a) = -g(b)f(a) = h(b)$. It follows that we may apply Rolle's theorem to conclude that there exists some $x \in (a, b)$ such that $h'(x) = 0$. Thus:

$$h'(x) = g'(x)f(x) + f'(x)g(x) - f(a)g'(x) - g(b)f'(x) = 0$$

which implies that:

$$g'(x)[f(x) - f(a)] = f'(x)[g(b) - g(x)]$$

so:

$$\frac{f'(x)}{g'(x)} = \frac{f(x) - f(a)}{g(b) - g(x)}$$

and the proof is complete. □

Problem 11.54a. Suppose that f is differentiable on $[a, b]$. Prove that if the minimum of f on $[a, b]$ is at a, then $f'(a) \geq 0$, and if it is at b, then $f'(b) \leq 0$.

Proof. Suppose that a is the minimum and $f'(a) < 0$. If we let $\epsilon = |f'(a)|$, then we note that we can choose a δ such that for all x with $a < x < a + \delta$ such that:

$$\frac{f(x) - f(a)}{x - a} < 0 \implies f(x) < f(a)$$

which contradicts the fact that a is a minimum. Thus, we must have $f'(a) \geq 0$. We proceed in a similar fashion to show that $f'(b) \leq 0$. □

Problem 11.54b. Suppose that $f'(a) < 0$ and $f'(b) > 0$. Show that $f'(x) = 0$ for some x in (a, b).

Proof. Clearly, from above, the minimum can't be at a or b, so it must be in (a, b). Therefore, at this point, we will have $f'(x) = 0$. □

Problem 11.54c. We let $g(x) = f(x) - cx$. We note that $g'(x) = f'(x) - c$. Since $'f(a) < c$, then $g'(a) < 0$. In addition, $g'(b) > 0$. Therefore, there is a point x such that $g'(x) = f'(x) - c = 0$, so $f'(x) = c$.

Problem 11.56. If $|f|$ is differentiable at a and f is continuous at a, then f is differentiable at a.

Proof. First, consider the case where $f(a) > 0$. It follows that there exists an interval around a such that f is positive, as f is continuous. In other words, for all x in such an interval, we have $|f|(x) = |f(x)| = f(x)$. Thus $|f|'(x) = f(x)$.

Similarly, in the case of $f(a) < 0$, there exists an interval around a such that f is negative (as f is continuous). Thus, for all x in the interval, $f(x) = -|f|(x)$, so $f'(x) = -|f|'(x)$.

Finally, let us consider the case where $f(a) = 0$. We note that $|f|(a) = 0$. Thus, $|f|$ takes on a local minimum at a, and is differentiable, so by Fermat's theorem, we note that $|f|'(a) = 0$.

We then can easily apply the ϵ-δ definition to show that $f'(a) = 0$ as well. More specifically, we choose δ such that:

$$|x - a| < \delta \ \Rightarrow \ ||f|(a)| < \epsilon$$

and note that $||f|(a)| = |f(a)|$, so we can use this same δ to show that $f'(a) = 0$. This completes the proof. $\square$

Problem 11.57. For even n, $x^n + y^n = (x + y)^n$ only when $x = 0$.

Proof. Assume that for some even n, there is some x_0 such that $x_0^n + y^n = (x_0 + y)^n$. Without loss of generality, we assume that $x_0 > 0$. It follows that we can apply Rolle's theorem to the interval $[0, \ x_0]$ on $f(x) = x^n + y^n - (x + y)^n$ to conclude that there is a point x_1 such that:

$$f'(x_1) = nx_1^{n-1} - n(x_1 + y)^{n-1} = 0 \ \Rightarrow \ x_1^{n-1} = (x_1 + y)^{n-1}$$

Since n is even, then $n - 1$ must be odd, so the $n - 1$-th root of both sides of this equation must be positive. This implies that $x_1 = x_1 + y$, so $y = 0$, a contradiction. $\square$

Problem 11.58. If n is even and $f(x) = x^n$, then every tangent line intersects $f(x)$ only once.

Proof. Any tangent line to $f(x)$ at $x = a$ is clearly of the form:

$$y(x) = f'(a)(x - a) + f(a) = na^{n-1}(x - a) + a^n = nxa^{n-1} + a^n(1 - n)$$

Let x_0 be a point such that $y(x_0) = f(x_0)$. Without loss of generality, we assume that $x_0 > a$. We apply Rolle's theorem to $g(x) = y(x) - f(x)$ to conclude that there is a point x in $(a, \ x_0)$ such that $g'(x) = 0$, so:

$$g'(x) = y'(x) - f'(x) = na^{n-1} - nx^{n-1} = 0 \ \Rightarrow \ a^{n-1} = x^{n-1}$$

As we concluded above, since $n - 1$ is odd, we therefore must have $a = x$, a contradiction. Thus, the tangent line can only intersect $f(x)$ at a. $\square$

Problem 11. If f' is increasing, then every tangent line intersects f only once

Proof. Once again, any tangent line to f will be of the form:

$$y(x) = f'(a)(x - a) + f(a)$$

Let us assume that there exists some point x_0 such that $f(x_0) = y(x_0)$, where $x_0 \neq a$. As we did above, we apply Rolle's theorem to the function $g(x) = f(x) - y(x)$ to the interval $(a, \ x_0)$ (we assume without loss of generality that $x_0 > a$, but the proof is identical for $x_0 < a$).

There must be a point x such that $g'(x) = 0$. Thus:

$$g'(x) = f'(x) - f'(a) = 0 \ \Rightarrow \ f'(x) = f'(a)$$

However, we know that f' is increasing, so we must have $f'(a) < f'(x)$, a contradiction. It follows that no such point x_0 exists, and the tangent line intersects f only once. $\square$

Problem 11.60. Suppose that $f(0) = 0$ and $f'(x)$ is increasing. Prove that the function $g(x) = f(x)/x$ is increasing on $(0, \ \infty)$.

Proof. Consider:

$$g'(x) = \frac{f'(x)}{x} - \frac{f(x)}{x^2}$$

Let us consider some point $y \in (0, \infty)$. We note from the mean value theorem that there exists some point $z \in (0, y)$ such that:

$$f'(z) = \frac{f(y)}{y}$$

which implies that:

$$\frac{f'(z)}{y} = \frac{f(y)}{y^2}$$

Since f' is increasing and $z < y$, it follows that $f'(z) < f'(y)$, so we may conclude that:

$$\frac{f(y)}{y^2} = \frac{f'(z)}{y} < \frac{f'(y)}{y} \implies g'(y) = \frac{f'(y)}{y} - \frac{f(y)}{y^2} > 0$$

Therefore, for any point $y \in (0, \infty)$, we will have $g'(y) > 0$, so $g(x) = f(x)/x$ is increasing, and the proof is complete. $\qquad\square$

Problem 11.61. Use derivatives to prove that if $n > 1$, then:

$$(1 + x)^n > 1 + nx$$

for $-1 < x < 0$ and $0 < x$.

Proof. Assume that there exists some point y such that $-1 < y < 0$ or $0 < y$ and $(1 + y)^n < 1 + ny$.

We let $f(x) = (1 + x)^n$. Note that $f(x) > 0$ for all x that we consider In the case of $0 < y$, we can use mean value theorem to conclude that there exists some point z such that:

$$f'(z) = \frac{f(y) - f(0)}{y} < \frac{ny + 1 - 1}{y} = n$$

but this clearly cannot be the case, as $f'(x) = n(1 + x)^{n-1}$, and $(1 + x)^{n-1} > 1$ for $x > 0$. For the case of $-1 < y < 0$, we can again use mean value theorem to conclude that there must be a point z such that:

$$f'(z) = \frac{f(0) - f(y)}{-y} > \frac{1 - 1 - ny}{-y} = n$$

But again, this clearly cannot be the case as $f'(x) = n(1 + x)^{n-1}$ and $(1 + x)^{n-1} < 1$ for all x where $-1 < x < 0$.

Thus, we must have $(1 + x)^n > 1 + nx$ for all $x > 0$ and x with $-1 < x < 0$. This completes the proof. $\qquad\square$

Problem 11.63a. Prove that if $f'(a) > 0$, and f' is continuous at a, then f is increasing on some interval containing a.

Proof. Since $f'(a) > 0$, we let it equal ϵ. We then note that by definition of continuity, we maay choose some δ such that:

$$|x - a| < \delta \implies |f'(x) - f'(a)| < \epsilon = f'(a)$$

or, in other words, given some $x \in U = (a - \delta, a + \delta)$, we will have $f'(x) > 0$. It follows by definition of the derivative that f is increasing on U and the proof is complete. $\qquad\square$

Problem 11.63b.

Problem 11.65a. Suppose that f is continuous on $[0, 1]$ and that f is increasing at a for every $a \in [0, 1]$. Prove that f is increasing on $[0, 1]$.

Proof. Consider some point b such that $0 < b < 1$. Let us consider the subset $[b, \ 1]$. Since f is continuous, it takes on its maximum and minimum values.

Clearly, f cannot take on its minimum value at some $c > b$, as we know that f is increasing at c, so for some δ (which does exist), we note that $f(x) < f(c)$ when $c - \delta < x < c$, which contradicts the fact that f takes on its minimum at c.

Thus, f must take on its minimum value at b. It follows that given some x, and some $y > x$, we will have $f(x) < f(y)$, as $y \in [x, \ 1]$. It follows by definition that f is increasing on $[0, \ 1]$ and the proof is complete. $\qquad\square$

Problem 11.65b. Prove the above theorem without the assumption that f is continuous.

Proof. Let:

$$S_b = \{x \ : \ f(y) \geq f(b) \text{ for all } y \in [b, \ x]\}$$

Clearly, such a set has an upper bound ($x = 1$), so it must have a least upper bound (a supremum), which we call s. Assume that $s < 1$.

By definition, there exists some δ such that for z with $s < z < s + \delta$, then $f(s) < f(z)$. In addition, we can choose such a δ such that $s + \delta < 1$. We then choose some y such that $s - \delta < y < s$, and note that $y \in S_b$, so:

$$f(b) \leq f(y) < f(s) < f(z)$$

so $z \in S_b$, contradicting the fact that s is the least upper bound. It follows that $s = 1$. Finally, we choose y such that $1 - \delta < y < 1$, and note that $y \in S_b$, so $f(b) \leq f(y) < f(1)$, so 1 is also in S_b. Thus, $S_b = [b, \ 1]$.

Finally, we pick x and y such that $x < y$. We note that $y \in [x, \ 1] = S_x$, so $f(x) < f(y)$.

$\qquad\square$

Problem 11.65c. Prove that if f is increasing at a, and f is differentiable at a, then $f'(a) \geq 0$.

Proof. Note that by definition, there is an interval around a, such that for x in the interval, $x < a$ implies that $f(x) < f(a)$ and $x > a$ implies that $f(a) > f(x)$.

It follows that:

$$g(x) = \frac{f(x) - f(a)}{x - a}$$

is always positive. Assume that $f'(a) < 0$. It follows that $f'(a) = \lim_{x \to a} g(x)$ must be greater than or equal to 0, as if this weren't the case, we simply let $\epsilon = |f'(a)|$ and derive a contradiction. $\qquad\square$

Problem 11.65d. Let's go back to the definition of the derivative. We know that if $f'(a)$ exists and is greater than 0, it follows that we can choose a δ such that:

$$|x - a| < \delta \ \Rightarrow \ \left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < f'(a)$$

Thus, for $x \in (a - \delta, \ a + \delta)$, we must have:

$$\frac{f(x) - f(a)}{x - a} > 0$$

In the case that $x < a$, so $x - a$ is negative, we will have $f(x) - f(a) < 0$, so $f(x) < f(a)$. In the case that $a < x$, so $x - a$ is positive, we will have $0 < f(x) - f(a)$, so $f(a) < f(x)$.

To summarize, for x such that $a < x < a + \delta$, we have $f(a) < f(x)$ and for x such that $a - \delta < x < a$, we have $f(x) < f(a)$. It follows, by definition, that f is increasing at a.

Problem 11.65e. Show that if f is continuous on $[0, \ 1]$, and $f'(a) > 0$ for all $a \in [0, \ 1]$, then f is increasing on $[0, \ 1]$.

Proof. Since $f'(a) > 0$, we can use the previous problem's result to conclude that f is increasing at a, for all $a \in [0, 1]$. Then, using Problem 11.65a, we note that this implies that f is increasing on $[0, 1]$. $\qquad \square$

Problem 11.65f. Suppose that f is continuous on $[0, 1]$ and $f'(a) = 0$ for all $a \in (0, 1)$. Apply the result in 11.65e to the function $g(x) = f(x) + \epsilon x$ to show that $f(1) - f(0) > -\epsilon$.

Similarly, show that $f(1) - f(0) < \epsilon$ be considering $\epsilon x - f(x)$. Conclude that $f(0) = f(1)$.

Proof. Clearly, g is continuous on $[0, 1]$, as it is the sum of two continuous functions. We also note that $g'(x) = f'(x) + \epsilon = \epsilon$, so $g'(x) > 0$ for all $x \in [0, 1]$ (we assume that ϵ is positive). It follows that g is increasing on $[0, 1]$. It follows by defintion that $g(1) > g(0)$, so we have:

$$0 < g(1) - g(0) = [f(1) + \epsilon] - f(0) \implies -\epsilon < f(1) - f(0)$$

We proceed in an identical fashion using the function $h(x) = \epsilon x - f(x)$ to show that $f(1) - f(0) < \epsilon$.

Now, assume that $f(1) - f(0) = c$, where $c > 0$. We let $\epsilon = c$ and derive a contradiction. We can derive a similar contradiction for $c < 0$. Thus, we must have $f(1) - f(0) = 0$, and the proof is complete. $\qquad \square$

9. OTHER PROOFS

Proposition 3. *If $f : A \to \mathbb{R}$ is differentiable at a, and $g : B \to \mathbb{R}$ is differewntiable at $b = f(a)$, with $f(A) \subset B$. Then $g \circ f$ is differentiable at a with:*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$$

Proof. First, we define two functions that effectively act as the "derivatives" of f and g for x approaching a and $b = f(a)$ respectively:

$$(1) \qquad \tilde{f}(x) = \begin{cases} \frac{f(x) - f(a)}{x - a} & \text{if } x \neq a \\ f'(a) & \text{if } x = a \end{cases}$$

$$(2) \qquad \tilde{g}(x) = \begin{cases} \frac{g(x) - g(b)}{x - b} & \text{if } x \neq b \\ g'(b) & \text{if } x = b \end{cases}$$

Clearly, by definition of the limit and the derivative, these functions are continuous at a and b respectively.

We assert that:

$$\frac{g(f(x)) - g(f(a))}{x - a} = \tilde{g}(f(x))\tilde{f}(x)$$

for all $x \neq a$, which can easily be verified by checking the cases of $f(x) = f(a)$ and $f(x) \neq f(a)$. The main idea here is that we are showing the expression of which we are calculating the limit is equal to the product of expressions which one would take the limit of to get $g'(f(a))$ and $f'(a)$.

From here, it is just a matter of actually carrying out the limit:

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} \tilde{g}(f(x))\tilde{f}(x) = \lim_{x \to a} \tilde{g}(f(x)) \lim_{x \to a} \tilde{f}(x) = \tilde{g}(b)f'(a) = g'(f(a))f'(a)$$

and the proof is complete. $\qquad \square$

Proposition 4. *If $f : [a, b] \to \mathbb{R}$ is continuous and is increasing on (a, b), then f is increasing on $[a, b]$.*

Proof. Assume that there exists some $x \in [a,\ b]$ such that $f(b) \leq f(x)$. It follows that since f is increasing, we can choose $y \in (x,\ b)$ such that $f(b) \leq f(x) < f(y)$.

We then apply intermediate value theorem to the interval $[y,\ b]$. The function f is continuous on this interval, so it follows that given c such that $f(b) < c < f(y)$, there must be some $z \in (y,\ b)$ such that $f(z) = c$. But this is a contradiction, as this would imply f is not increasing on $(a,\ b)$.

We make the same argument for the other side of the interval. $\square$